Algebraic Singularity Method for Mass Measurement with Missing Energy

Ian-Woo Kim

University of Wisconsin-Madison

work in progress

DM,LHC and Cosmology

KIAS

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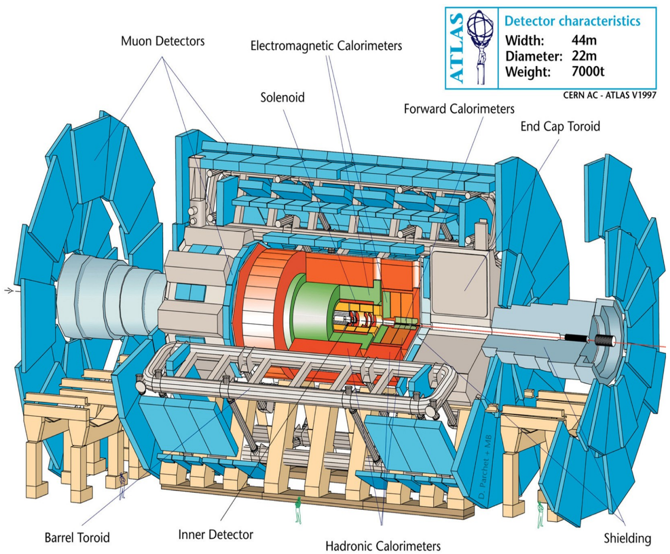
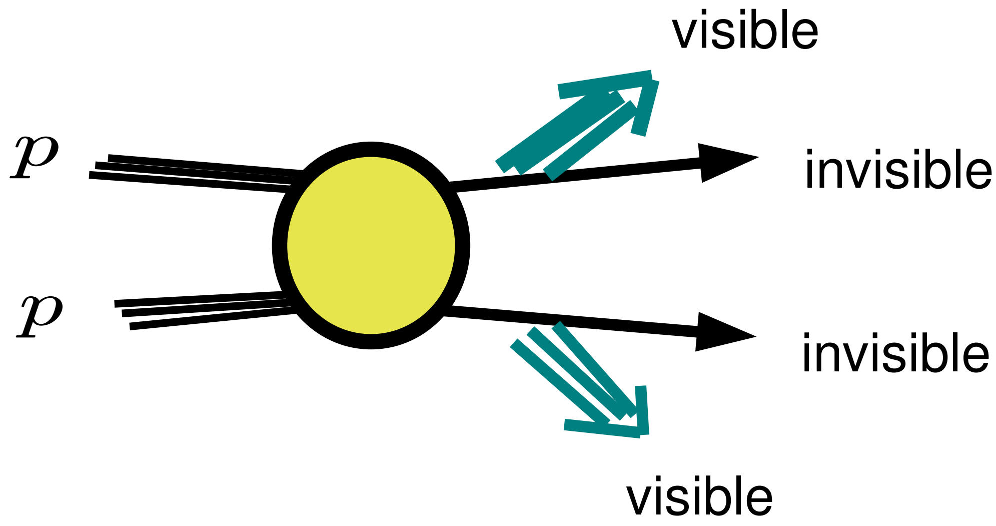


Prepare for discovery of Dark Matter particle!

Mass Measurement with a pair of missing energy particles is important.



Reconstructable event : Solve mass shell equation for each event

Nojiri, Polesello, Tovey (2003)

Cheng, Gunion, Han, Marandella, McElrath (2007)

Nonreconstructable event :

Use end point or cusps of kinematic variable

Hinchliffe, Paige, Shapiro, Soderqvist, Yao (1997)

Han, IWK, Song (2009)

Transverse mass variables

Lester, Summers (1999)

Cho, Choi, Kim, Park (2007)

Bar, Gripaios, Lester (2007)

General Description of Mass Measurement

Bayes' theorem

$$P(\mathrm{th}|\mathrm{exp}) \propto P(\mathrm{exp}|\mathrm{th})$$

Quantum Field Theory

$$P(\mathrm{exp}|\mathrm{th}) = P_{\mathrm{exp}} \otimes P_{\mathrm{QFT}}$$

$$P_{\mathrm{QFT}} = \int d\mathrm{PS} \otimes |\mathcal{M}|^2$$

Assume $|\mathcal{M}|^2$ is smooth.

$$\int d\mathrm{PS} = \int \ldots \int d^4 p_1 \ldots \int d^4 p_n \delta(p^2 - m^2) \ldots$$

Phase space is defined as a submanifold in total momentum space.

$|\mathcal{M}|^2$ has dependence on all the couplings, while PS depends only on masses.

Global analysis : obtain probabililty distribution with $\int d\mathrm{PS}$ and $|\mathcal{M}|^2$

Ultimate method, but depend on too many parameters.

Usually, rely on Monte Carlo template technique.

Kinematic analysis : using $\int d\mathrm{PS}$ only

Model independent, using only essential parameters

cannot use event profile since we ignore $|\mathcal{M}|^2$

⇓

Reconstructable: check whether each event satisfies mass shell equations

$$(\mathrm{likelihood}) \propto P_{\mathrm{ev1}} \times P_{\mathrm{ev2}} \times \dots$$

Nonreconstructable: we cannot determine whether each event is on PS or not.

But special points in kinematic distribution can appear

⇒ Kinematic Singularity point

(non-smooth point in a distribution)

Singularity in Distribution of Kinematic Variables

Explicit kinematic variable : no dependence on trial mass parameters

→ Obtain relations between mass parameters directly

e.g. invariant mass, angular variable, PT distribution

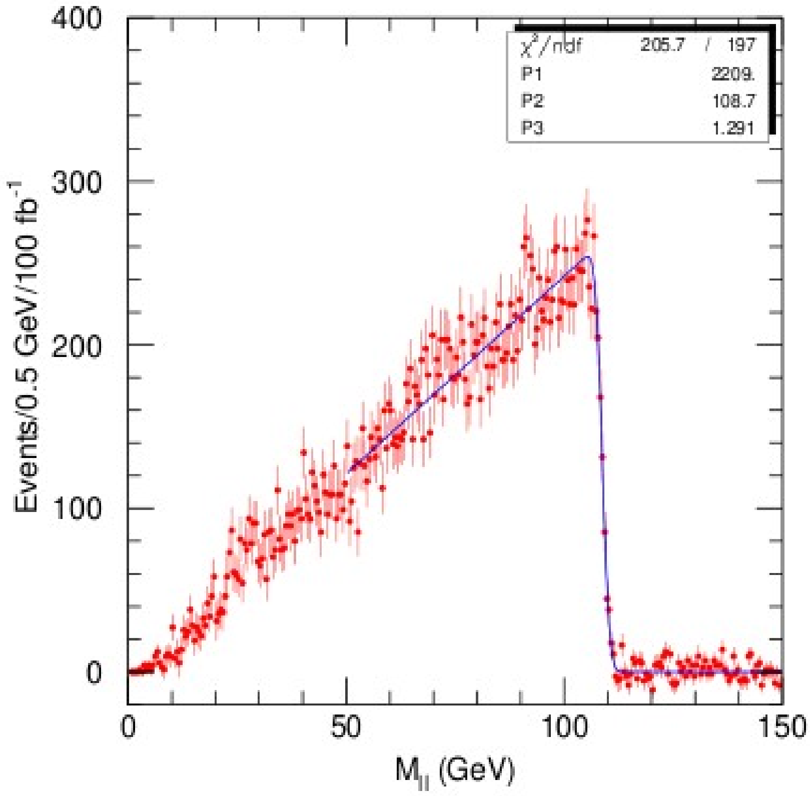
Implicit kinematic variable : depends on trial mass parameters

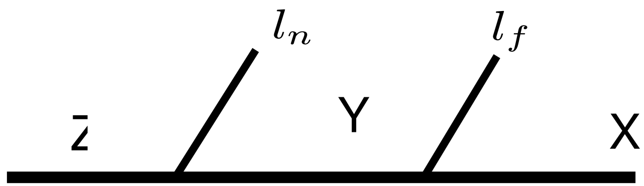
→ Obtain relations between mass parameters at a trial position

e.g. MT2 (mixed kinematic variable)

I will construct a new totally implicit kinematic variable in the following.

End point singularity in invariant mass distribution



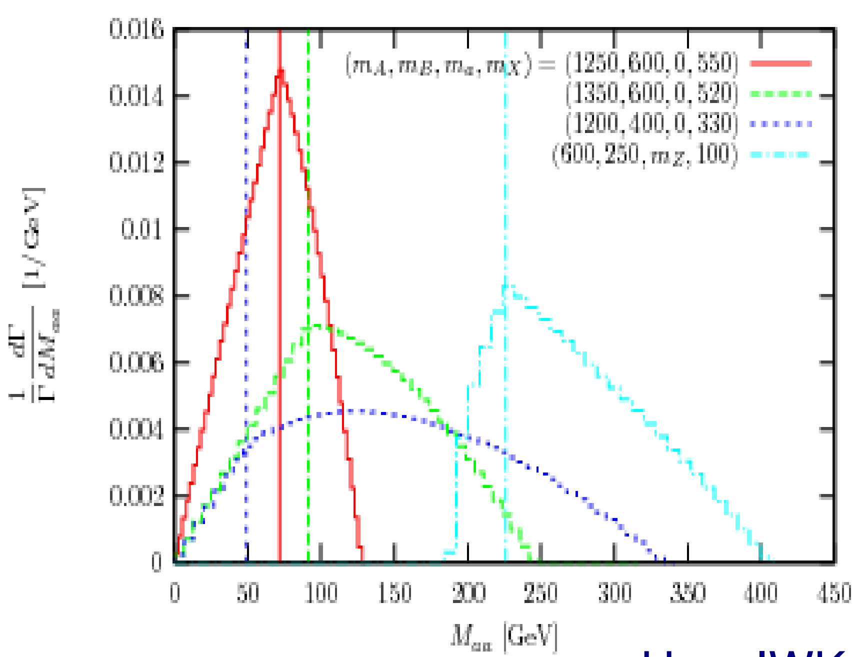


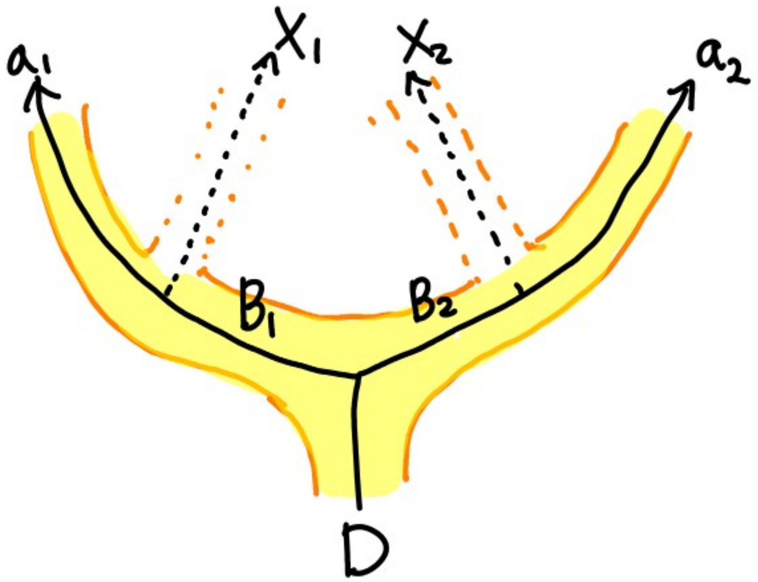
$$m_{ll} = \sqrt{(l_n + l_f)^\mu (l_n + l_f)_\mu}$$

from Bachacou, Hinchliffe, Paige (1999)

$$m_{ll}^{\text{edge}} = \sqrt{\frac{(m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)}{m_Y^2}}$$

Cusp and End point singularity in invariant mass distribution



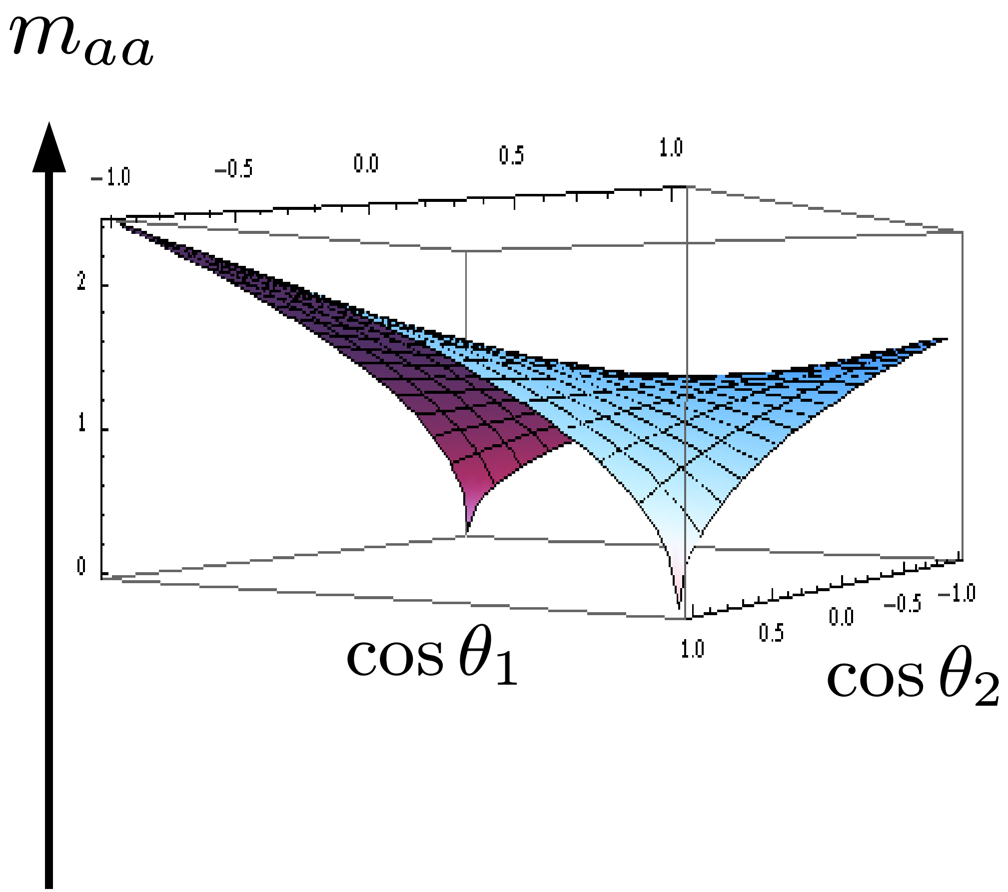


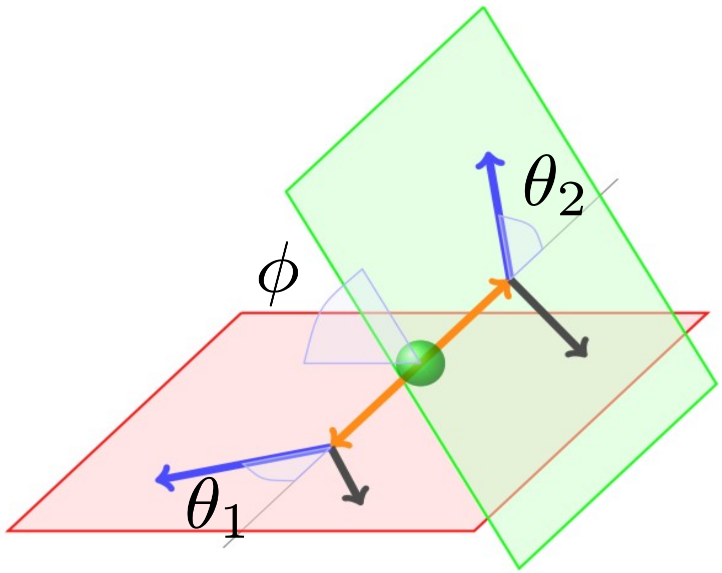
Han, IWK, Song (2009)

$$\frac{M_{aa}^{\mathrm{cusp}}}{M_{aa}^{\mathrm{max}}} = \exp(-2\eta) = \frac{m_D^2 - 2m_B^2}{2m_B^2} - \frac{m_D}{m_B}\sqrt{\frac{m_D^2}{4m_B^2} - 1}$$

$$M_{aa}^{\mathrm{cusp}} M_{aa}^{\mathrm{max}} = m_B^2 \left(1 - \frac{m_X^2}{m_B^2}\right)^2$$

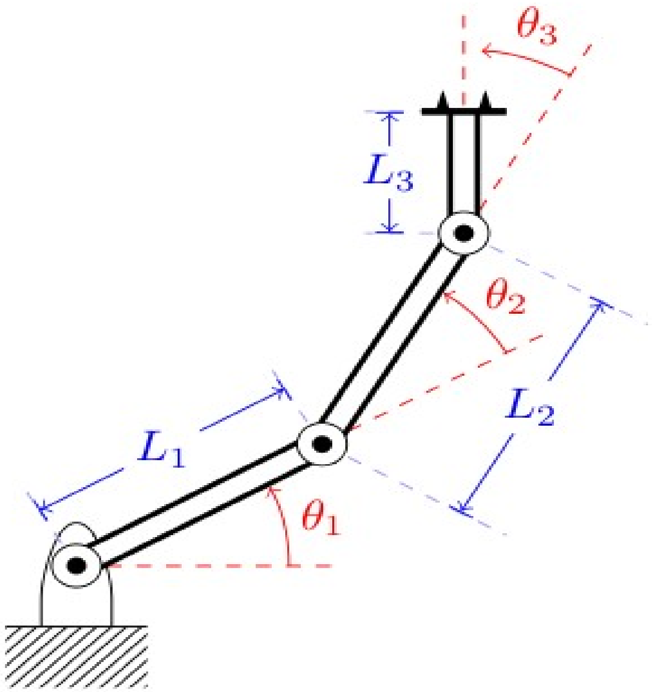
Why end point and cusp appear?





Phase Space is folded for a kinematic variable.

Analogy with Robotics



Position of the hand = Event Observable

Angles at each joint = Phase space variable

Arm Length = Model parameters

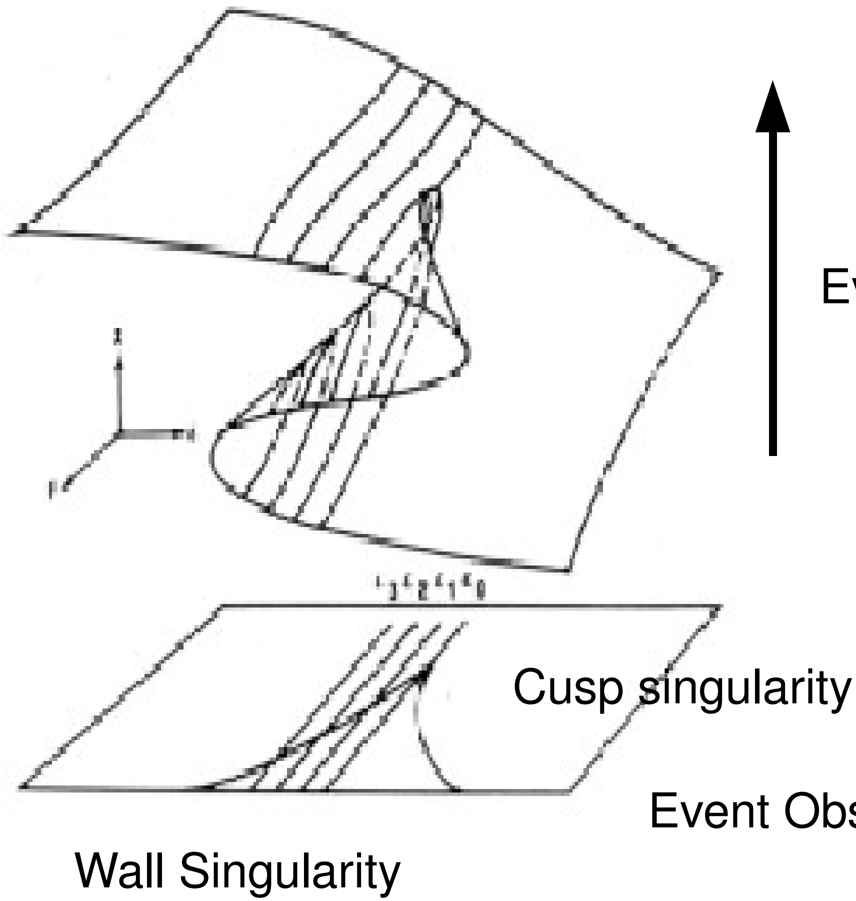
Although joint angle space $S^1 \times S^1 \times S^1$ has no special point, the resultant hand movement can show some singular behavior.

Dimension of local tangent space of the movement changes!

$$f : \mathcal{J} \to \mathcal{C}$$

Jacobian matrix $\left(\frac{\partial c_m}{\partial j_n} \right)$ has a reduced rank at the singular point.

General Mathematical Method for Singularity



Event Observable space = projected image of PS.

$$\int d\mathrm{PS} = \int \ldots \int d^4p \ldots \delta(g_1)\delta(g_2) \ldots$$

Phase space is defined by solution space of coupled polynomial equations.

$$\begin{aligned} g_1 &= 0 \\ g_2 &= 0 \\ g_3 &= 0 \\ &\vdots \end{aligned} \quad \longrightarrow \quad \text{affine variety}$$

x : event unknowns

q : event observables

At a singularity point in event observable space,

Jacobian $\left(\dfrac{\partial g_i}{\partial x_j}\right)$ has a reduced rank

Dimension of Singularity
in Event Observable space and Event Unknown space

total dimension of momentum affine space : $D_{\mathrm{PS}} = N_{\mathrm{obs}} + N_{\mathrm{unknown}}$

# of Polynomial equations w.r.t. event unknowns : C_{unknown}

# of Polynomial equaitons indep of event unknowns: $C_{\mathrm{observable}}$

Dimension of Phase space : $D_{\mathrm{PS}} - C_{\mathrm{observable}} - C_{\mathrm{unknown}}$

Rank of $\frac{\partial g_i}{\partial x_j}$ at the ordinary point : C_{unknown}

degree of singularity = reduced amount of rank of $\frac{\partial g_i}{\partial x_j}$: $\deg$

Dimension of Phase space purely immersed in unknown space at a given point

$$N_{\mathrm{unknown}} - C_{\mathrm{unknown}} + \deg$$

Groebner basis:

With lexicographic ordering $x_1 > x_2 > x_3 > x_4 > \ldots$

$$g_1(x_1, x_2, x_3, x_4, \ldots) = 0$$
$$g_2(x_2, x_3, x_4, \ldots) = 0$$
$$g_3(x_3, x_4, \ldots) = 0$$
$$g_4(x_4, \ldots) = 0$$

Equations a re sequentially solvable.

Jacobian matrix has upper triangular form.

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} X & \ldots & \ldots & \\ & X & X & \ldots \\ & & X & \ldots \end{pmatrix}$$

Vanishing diagonal component is necessary in this basis for a reduced rank of Jacobian.

For the row vectors of Jacobian with vanishing diagonal component, we can directly check whether they are linearly dependent or not.

$$\vec{v}_1 = (a_1, a_2, a_3, \ldots)$$

$$\vec{v}_2 = (b_1, b_2, b_3, \ldots)$$

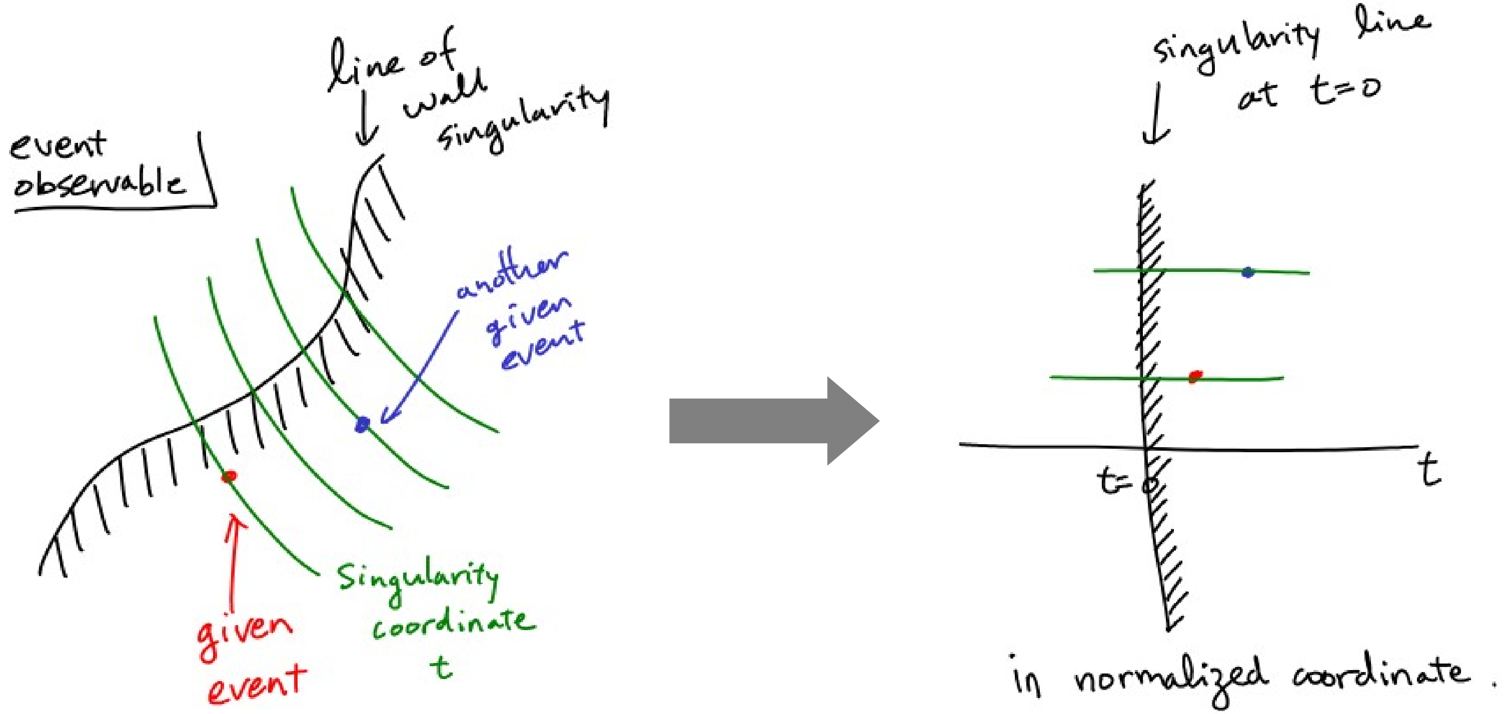
$$\vec{v}_1 \parallel \vec{v}_2 \quad \text{if} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots$$

By this way, we can classify all the singularities in event observables.

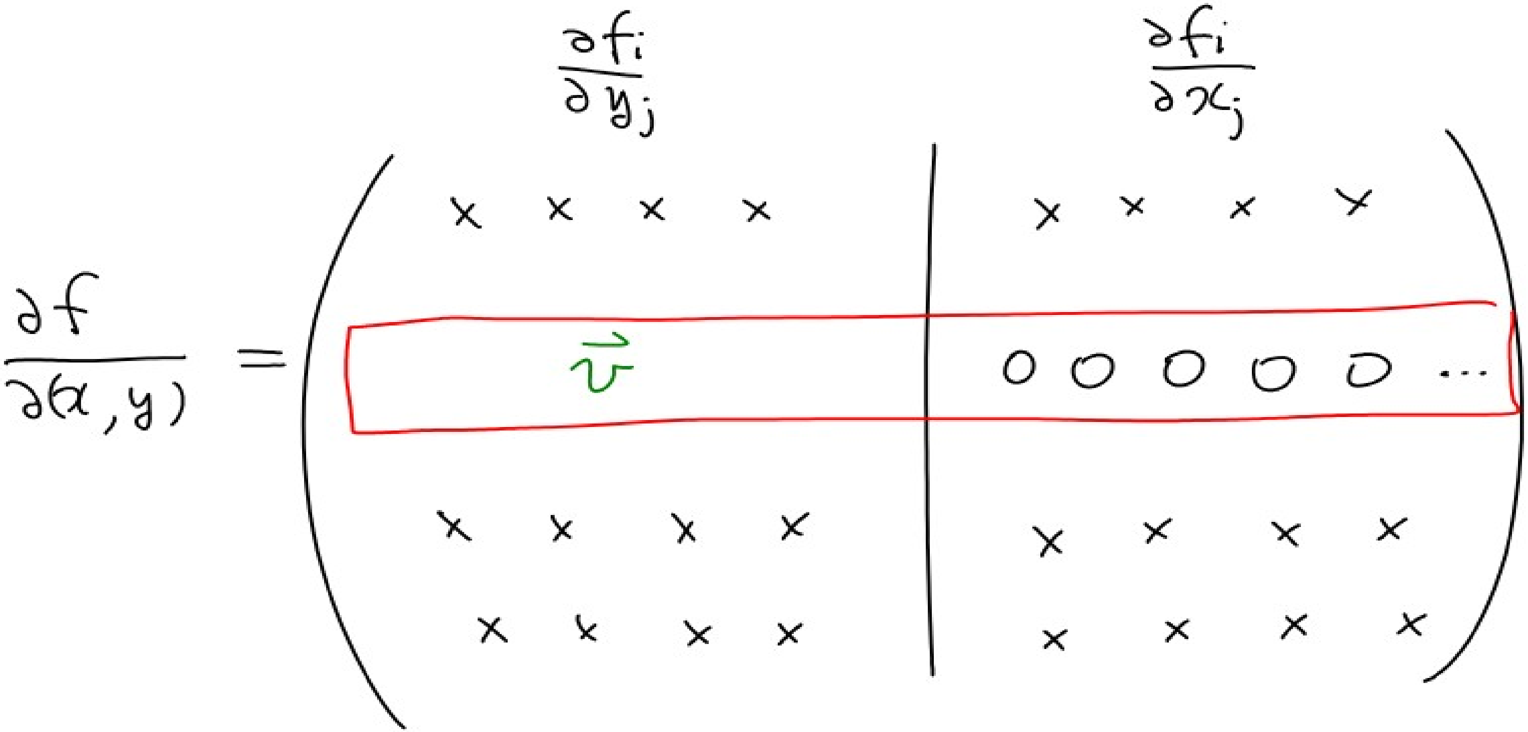
Singularity Coordinate

Once we know where a singularity is located, we need to define a normalized coordinate near the singularity point.

Project all the event points near the singularity on the coordinate.

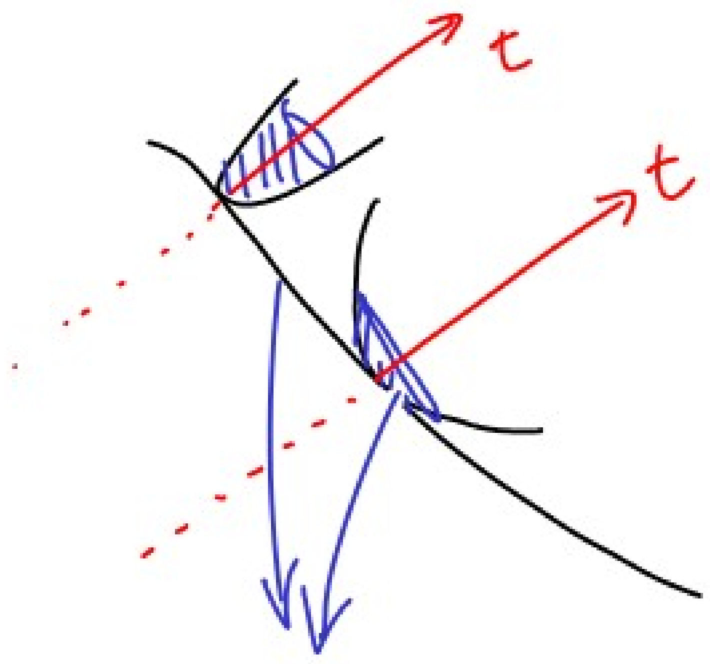


Singularity Coordinate direction is already determined by the reduced rank condition.



$\vec{v}$ is tangent direction for singularity coordinate, which is normal to singularity plane.
I will call it normal direction of singularity.

To determine the scaling of coordinate, we choose equal density normalization.



Local description of phase space is useful for this procedure : second fundamental form

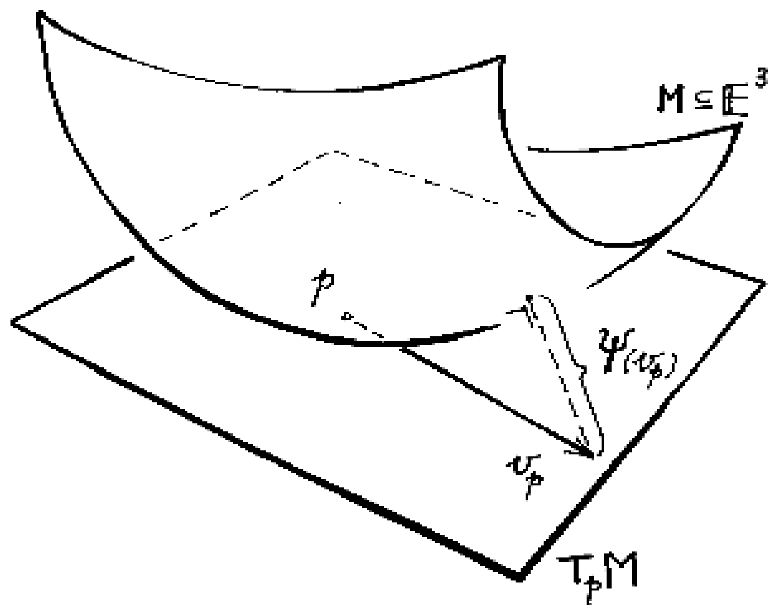
Same event unknown volume give rise to the same singularity coordinate.

Using Second Fundamental Form:

Describe 2nd order discrepancy between tangential plane and actual hypersurface.

$$I\!I(v, w) = (\nabla_v w)^\perp$$

which shows the information of the direction of curvature of the hypersurface.



Second Fundamental Form of Algebraic Variety

$$g_i = 0 \quad \Rightarrow \quad g_i + \frac{\partial g_i}{\partial y_j} dy^j + \frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k = 0$$

local variation

dy is confined to the normal space and dx is confined to the tangential space.

$$\frac{\partial g_i}{\partial y_j} dy^j = -\frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k$$

defines a map : $TM_p \times TM_p \to NM_p$

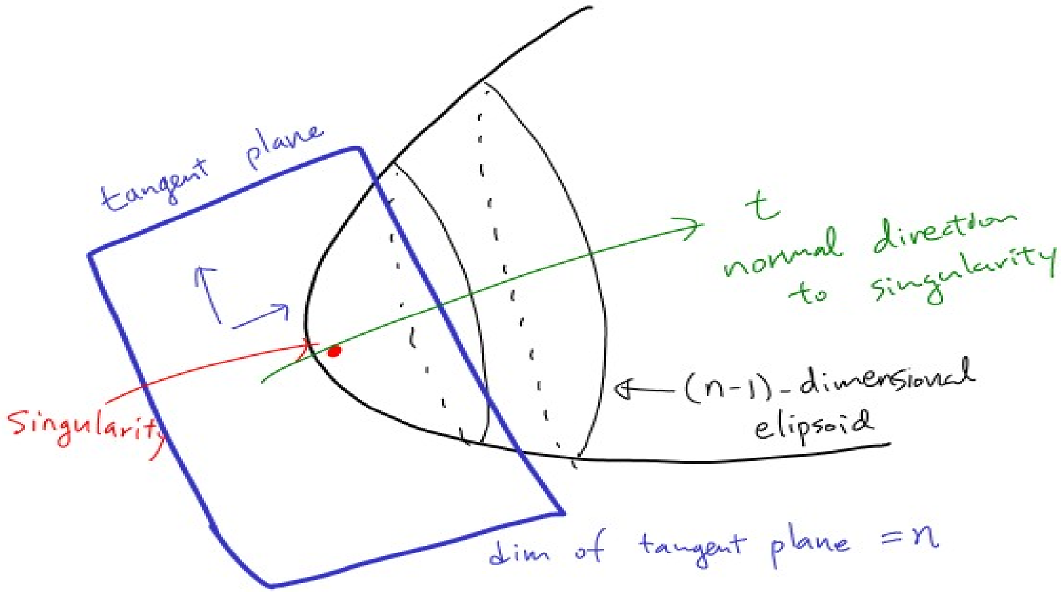
Normalization and Local scaling behavior near singularity

If we define the linear singularity coordinate using $\vec{v}$ and 2nd fundamental form

$$dt = \vec{v} \cdot I\!I$$

The linear coordinate is characterized by $n \times n$ matrix M, where n is the dimension of tangent plane immersed completely along event unknown space.

We have to choose scale factor of t so to have the same weight for each event. Event unknown volume is the guideline for this.



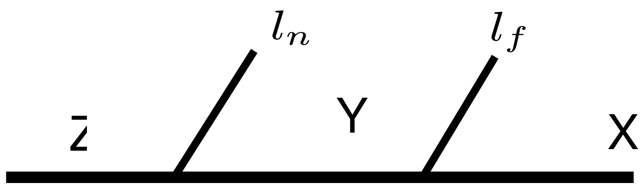
Locally,

$$t = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2}$$

$$(\mathrm{vol}) \propto r^n \sim \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{n/2} t^{n/2}$$

$$\frac{d\Gamma}{dt} \propto (a_1^2 + \cdots + a_n^2)^{\frac{n}{2}} t^{\frac{n}{2}-1} \frac{d\Gamma}{d(\mathrm{vol})}$$

Reanalysis of Invariant mass in cascade decay



Kinematic constraint equations:

$$X^2 = m_X^2$$
$$(X + l_f)^2 = m_Y^2$$
$$(X + l_f + l_n)^2 = m_Z^2$$

Event unknowns :

$$X^\mu = (X_0, X_1, X_2, X_3)$$

Event Observable : $l_n^\mu \quad l_f^\mu$

Coordinate in C.M. frame of l_n and l_f

$$l_n'^{\mu} = \left(\frac{E_{\rm cm}}{2}, 0, 0, \frac{E_{\rm cm}}{2} \right) \qquad l_f'^{\mu} = \left(\frac{E_{\rm cm}}{2}, 0, 0, -\frac{E_{\rm cm}}{2} \right)$$

Groebner basis : $X_0' > X_3' > X_1' > X_2'$

$$g_1 = 2E_{\rm cm}X_0' + (E_{\rm cm}^2 + m_X^2 - m_Z^2) = 0$$

$$g_2 = 2E_{\rm cm}X_3' + E_{\rm cm}^2 + 2m_Y^2 - m_X^2 - m_Z^2 = 0$$

$$g_3 = E_{\rm cm}^2 X_1'^2 + E_{\rm cm}^2 X_2'^2 + E_{\rm cm}^2 m_Y^2 - (m_Z^2 - m_Y^2)(m_Y^2 - m_X^2) = 0$$

Jacobian matrix for Groebner basis:

$$\left(\frac{\partial g_i}{\partial X_j'} \right) = \begin{pmatrix} 2E_{\rm cm} & 0 & 0 & 0 \\ & 2E_{\rm cm} & 0 & 0 \\ & & E_{\rm cm}^2 X_1' & E_{\rm cm}^2 X_2' \end{pmatrix}$$

Nontrivial reduced rank Jacobian can arise when

$$X_1' = X_2' = 0$$

Then, $$E_{\rm cm}^2 = \frac{(m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)}{m_Y^2} \equiv E_{\rm cm0}^2$$ by $$g_3 = 0$$

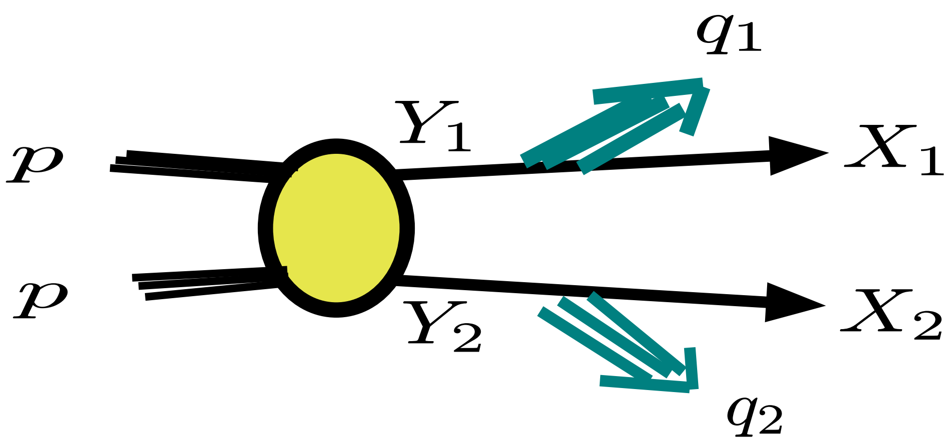
Perpendicular direction to tangent space at the singularity: rather trivial

$$\frac{\partial g_3}{\partial E_{\rm cm}} = 2m_Y^2 E_{\rm cm}$$

Normalized coordinate:

$$t \equiv 4\pi m_Y^2 \left(\frac{E_{\rm cm0}^2 - E_{\rm cm}^2}{E_{\rm cm0}^2} \right)$$

Double Missing Particle Chain Topology:



Kinematic Constraint Equations :

$$X_1^2 = m_X^2 \qquad X_2^2 = m_X^2$$

$$(X_1 + q_1)^2 = m_Y^2 \qquad (X_2 + q_2)^2 = m_Y^2$$

$$\vec{X}_{1T} + \vec{X}_{2T} = \vec{P}_T^{\rm miss}$$

Event unknowns:

$$X_1^\mu = (X_{10}, X_{11}, X_{12}, X_{13}) \qquad X_2^\mu = (X_{20}, X_{21}, X_{22}, X_{23})$$

Event observables:

$$q_1^\mu = (q_{10}, q_{11}, q_{12}, q_{13}) \quad q_2^\mu = (q_{20}, q_{21}, q_{22}, q_{23}) \quad \vec{P}_T^{\rm miss} = (P_{T1}, P_{T2})$$

Groebner basis : $X_{10} > X_{13} > X_{20} > X_{21} > X_{22} > X_{23} > X_{11} > X_{12}$

$$
\begin{array}{c}
g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6
\end{array}
\begin{pmatrix}
X & X & & & & & X & X \\
 & X & & & & & X & X \\
 & & X & & & X & X & X \\
 & & & 1 & & & 1 & \\
 & & & & 1 & & & 1 \\
 & & & & & X & X & X
\end{pmatrix}
$$

reduced rank condition :

$$2(q_{10}^2 - q_{13}^2)X_{13} - 2A_1 q_{13} = 0$$

$$2(q_{20}^2 - q_{23}^2)X_{23} - 2A_2 q_{23} = 0$$

$$\det \begin{pmatrix} \frac{\partial g_2}{\partial X_{11}} & \frac{\partial g_2}{\partial X_{12}} \\ \frac{\partial g_6}{\partial X_{11}} & \frac{\partial g_6}{\partial X_{12}} \end{pmatrix} = 0$$

These conditions give rise to the maximum MT2 condition and MAOS momentum.

Numerical analysis with singularity coordinate is now in progress. ⇨ Wait for our paper!

Conclusion

General description of mass measurement method leads us to investigate singularity structure in phase space.

We develop a new algebraic geometrical method for seeking for singularities in PS and find out tailored implicit variable for singularity.

This method naturally includes previous methods for nonreconstructable event topology such as end point and cusp in invariant mass and mT2. We can enhance such cases with this generalized method and generalize to different event topologies.